These formulas are quite similar to those obtained for evacuated cavities, and most of the comments made about the latter can be duplicated here, for example, that the electric currents should be parallel to the lines of the electric field to efficiently excite any given mode. We do not insist on this aspect, but notice that the formulas of (20) provide a purely formal solution of our problem. There remains the formidable task of actually determining the eigenvectors for any given geometry and disposition of gyrotropic material (in certain cases one might be satisfied with the structure of a single mode if operation at a resonant frequency is considered). This problem is not within the province of the present paper, and we shall only mention that cylindrical structures in the form of terminated waveguides are those for which the analysis can progress most satisfactorily.[9–12]

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An Impedance Transformation Method for Finding the Load Impedance of a Two-Port Network*

R. MITTRA†, MEMBER, IRE, AND R. J. KING‡, MEMBER, IRE

Summary—**An unknown load impedance terminating a lossy two-port junction can be calculated if the input impedance and junction parameters are known. It is to be shown that there exists a linear relationship, dependent upon two calibration constants, between the input reflection coefficient and a modified reflection coefficient of the load. Applying the linear transformation to the junction input impedance permits evaluation of the unknown load impedance. Calibration is accomplished by terminating the transmission line in at least three different reactances and measuring the corresponding input reflection coefficients. These data plot into the usual circular configuration on a Smith chart from which the necessary calibration data is obtained. When several load reactances are used, the calibration accuracy can be considerably increased, since the averaging advantage of plotting a mean straight line is utilized. Furthermore, once the junction has been calibrated, its equivalent T-network impedances and scattering coefficients may be found.**

I. INTRODUCTION

AN UNKNOWN load impedance which terminates a two-port junction can be calculated if the input impedance and parameters of the junction are known. Several methods are currently available for determining the network parameters, such as the three-point method, canonical method, and the scattering-matrix method.[1–10]

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† Engineering Experiment Station, University of Colorado, Boulder, Colo.; on leave of absence from University of Illinois, Urbana, Ill., during the summer of 1960.
‡ Engineering Experiment Station, University of Colorado, Boulder, Colo.; on leave of absence from Indiana Technical College, Fort Wayne, Ind.

[1] L. B. Felsen and A. A. Oliner, "Determination of equivalent circuit parameters for dissipative microwave structures," PROC. IRE, vol. 42, pp. 477–483; February, 1954.
[2] G. A. Deschamps, "Determination of reflection coefficient and insertion loss of a waveguide junction," *J. Appl. Phys.*, vol. 24, pp. 1046–1050; August, 1953.
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[9] M. H. Oliver, "Discontinuities in a concentric line impedance measuring apparatus," *Proc. IEE*, vol. 97, pt. 3, pp. 29–38; January, 1950. (A method suitable for lossless structures only.)
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This paper describes a procedure for finding the load impedance based on a graphical method (as are most other calibration procedures), but it avoids cumbersome complex number manipulations, the restriction that the junction be lossless, and large probable calibration error caused by insufficient data. When it is properly applied, it provides a means of evaluating load impedances directly, rapidly, and accurately.

It is to be shown that for a two-port junction there exists a linear relationship between the input reflection coefficient Γ_{in} and a modified load reflection-coefficient Γ_L' (yet to be defined). This relationship contains two calibration constants which are determined by means of a graphical calibration procedure; then applying this linear relationship to the network input impedance measurement permits evaluating the load impedance. The theory of this technique is described in the following.

II. General Theory[11]

The reciprocal, passive, two-port junction can be represented by the T-configuration of Fig. 1. With a load Z_L connected, the input impedance seen by a generator at port 1 is the usual

$$Z_{in} = Z_{11} - \frac{Z_{12}^2}{Z_{22} + Z_L}. \tag{1}$$

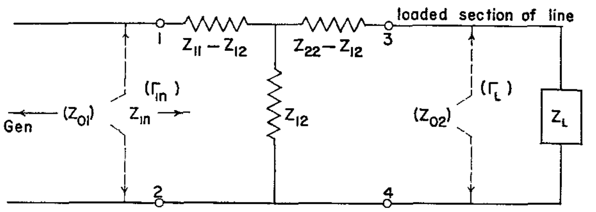


Fig. 1—Notation pertaining to a reciprocal, passive, two-port junction.

From the definition of reflection coefficient, one may express an input Γ_{in} for the T-network as[12]

$$\Gamma_{in} = \frac{Z_{in} - Z_{01}}{Z_{in} + Z_{01}} = \frac{Z_{11} - Z_{01}}{Z_{11} + Z_{01}} \left[\frac{Z_{22} + Z_L - \dfrac{Z_{12}^2}{Z_{11} - Z_{01}}}{Z_{22} + Z_L - \dfrac{Z_{12}^2}{Z_{11} + Z_{01}}} \right]. \tag{2}$$

Observe that (2) can be rearranged as

$$\Gamma_{in} - \frac{Z_{11} - Z_{01}}{Z_{11} + Z_{01}} = \frac{Z_{11} - Z_{01}}{Z_{11} + Z_{01}} \left[\frac{\dfrac{-2Z_{12}^2 Z_{01}}{Z_{11}^2 - Z_{01}^2}}{Z_{22} - \dfrac{Z_{12}^2}{Z_{11} + Z_{01}} + Z_L} \right]. \tag{3}$$

The quantity $Z_{22} - Z_{12}^2/(Z_{11} + Z_{01})$ in the denominator of (3), from comparison with (1), is the impedance looking into port 2 when the generator at port 1 is replaced by the matched load Z_{01}. This quantity is a fixed constant which, after normalization by Z_{02}, we designate as[13]

$$z_1 = \frac{1}{Z_{02}} \left(Z_{22} - \frac{Z_{12}^2}{Z_{11} + Z_{01}} \right) = r_1 + jx_1. \tag{4}$$

These parameters r_1, x_1 will appear later as the "calibration constants" mentioned before. Now define a normalized load impedance $z_L = Z_L/Z_{02} = r_L + jx_L$ so that substitution of z_1 and z_L into (3) and normalizing all the remaining terms by Z_{01} yields

$$\Gamma_{in} = \frac{z_{11} - 1}{z_{11} + 1} = \frac{z_{11} - 1}{z_{11} + 1} \left[\frac{\dfrac{-2z_{12}^2}{(z_{11}^2 - 1)r_1}}{1 + \dfrac{r_L}{r_1} + j\dfrac{x_L + x_1}{r_1}} \right], \tag{5}$$

where $z_{11} = Z_{11}/Z_{01}$ and $z_{12}^2 = Z_{12}^2/Z_{01}Z_{02}$. One should note that we have normalized the input impedance to the characteristic impedance of the input port Z_{01}, while the load impedance Z_L is normalized to the characteristic impedance of the output port Z_{02}.

Now simplify by making the substitutions

$$r_L' = \frac{r_L}{r_1}, \quad x_L' = \frac{x_L + x_1}{r_1}; \quad z_L' = r_L' + jx_L'. \tag{6}$$

Observe that we have defined a modified load impedance z_L' which is related to the actual load impedance z_L through parameters r_1, x_1. Correspondingly, define a "modified" load reflection coefficient Γ_L' related

[11] The general theory has been discussed by R. Mittra, "Impedance measurements through a discontinuity in a transmission line," 1955 IRE National Convention Record, vol. 3, pt. 8, pp. 85–91. See also "Impedance and Pattern Studies of Disk-Based Monopoles over Lossy Ground Survaces," Dept. of Elec. Engrg., Univ. of Colorado, Boulder, Scientific Rept. No. ERD-TN-60-763, AF Res. Div. Contract No. AF 19(604)-4556, pp. 57–78; July, 1960.

[12] Z_{01} and Z_{02} denote the characteristic impedances of the transmission lines connected to the input reference plane 1-2 and output reference plane 3-4 respectively as in Fig. 1.

[13] All normalized quantities appear in lower case letters.

to z_L' in the usual way:

$$\Gamma_L' = \frac{z_L' - 1}{z_L' + 1}. \tag{7}$$

Finally, substituting (7) into (5) and letting

$$\frac{z_{11} - 1}{z_{11} + 1} = \bar{a}$$

and

$$\frac{z_{11} - 1}{z_{11} + 1}\left[\frac{-z_{12}^2}{(z_{11}^2 - 1)r_1}\right] = \bar{b}, \tag{8}$$

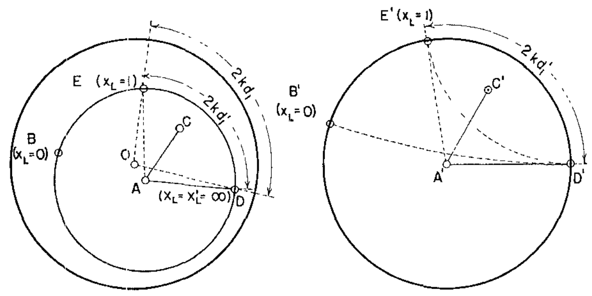
where $\bar{a}$ and $\bar{b}$ are complex network constants, puts (5) in the form

$$\Gamma_{in} - \bar{a} = \bar{b}(1 - \Gamma_L'). \tag{9}$$

Eq. (9) is a linear relationship between Γ_{in} and the output transformed reflection coefficient Γ_L'. The entire procedure for network-load impedance determination is based upon this relationship, and is described in the next sections.

sliding short which connects directly onto the output port. Assuming no losses in the shorted line, we have that this load reactance is the normalized $z_L = j \tan kd$, where d is the shorted transmission line length. An actual plot of the Γ_{in}-circle can then be made by means of a suitable impedance meter placed ahead of the network input terminals. In this manner, the circle typified in Fig. 2(a) is plotted point-for-point as the known load-reactance values are changed, with the center of the circle located at A.

The corresponding modified load reflection coefficient Γ_L' on the other hand will obviously plot on the periphery of the Γ_L' plane [shown by circle $D'E'B'$ in Fig. 2(b)], since from relation (6), the modified load resistance $r_L' = r_L/r_1$ is zero when the load is purely reactive. From the properties of a linear transformation[14] and from the known input impedance measurements, a point-to-point correspondence of the two circular loci of Γ_{in} and Γ_L' is established experimentally. Hence, distance $A'D'$ (to point $+1$) in Fig. 2(b) corresponds to a reactance termination provided by a quarter-wave



(a) (b)

Fig. 2—Typical linear transformation of points in Γ_{in} and $\Gamma_{L'}$ planes. (a) Γ_{in} plane. (b) $\Gamma_{L'}$ plane.

III. Principles of Network Calibration

We shall now demonstrate how the desired calibration measurements are made on the network, and how the unknown load impedance is then found. This straightforward procedure is based on the linearity of (9) and on the load impedance transformation relations (6). Some of the well-known properties of the Smith chart are repeated here in connection with the specific measurements involved in this procedure.

Suppose we consider two separate Smith charts on which are plotted the two reflection coefficients, Γ_{in} of (2) and Γ_L' of (7), which correspond to a given terminated junction. These are shown typically in Fig. 2. Fig. 2(a) represents the usual circular variation[10] of Γ_{in} (shown by circle DEB) obtainable when the output terminals are loaded with at least three values of reactance. These load reactances can be provided physically through the use of a uniform transmission line with a shorted line for which $x_L' = \infty$ from (6), since $x_L = \infty$, and so $\Gamma_L' = +1$. Then input-impedance measurements will provide a corresponding point D in the Γ_{in}-plane; this establishes distance OD in Fig. 2(a). The procedure is repeated with other load-reactance values to obtain other point-pairs and establish the circular loci of Fig. 2(a) with as much detail as may be desired. For every Γ_L' on the periphery of the Γ_L'-circle, there corresponds a certain x_L' and hence an x_L from (6). Thus, Γ_L' is related to x_L through the still unknown calibration constants r_1 and x_1. These are easily found by first establishing two independent equations involving unknowns r_1 and x_1; *e.g.*, refer to the expression for x_L' in (6) again, and denote by the symbols x_{L0}' and x_{L1}' as those x_L'-values which correspond to load re-

[14] The porperties of linear transformation are well known. It is an expansion and rotation followed by a translation. These operations leave all angles and ratios of distances invariant. See Appendix I for proof.

actances of $x_L=0$ and $x_L=1$, respectively. Then (6) yields

$$x_{L0}' = \frac{x_1}{r_1} \quad \text{and} \quad x_{L1}' = \frac{1+x_1}{r_1}. \tag{10}$$

Note that these two choices correspond to a short circuit placed first at the output port, and then moved one-eighth wavelength from the output. The input reflection coefficients Γ_{in} are then measured with these reactive loads in place, and plot typically as B and E in Fig. 2(a). Transformation to B' and E' in the Γ_L' plane follows from the linearity property. This permits the desired values of x_{L0}' and x_{L1}' to be read and inserted into (10) to yield the required x_1, r_1 values, completing the network calibration.

Suppose some unknown load impedance z_L is now connected to the junction, and an impedance meter is used to measure the input port reflection coefficient Γ_{in} which appears as point C in Fig. 2(a). This point maps into the Γ_L' plane as point C' in Fig. 2(b) (found from the equality of anles CAD and $C'A'D'$ and from equal magnifications CA/DA and $C'A'/D'A'$). But point C' also identifies an impedance z_L' (corresponding to that Γ_L'); finally, z_L is found from this z_L' by means of (6) and by use of the known r_1, x_1 values determined from the method already described.

IV. Theory of the Cotangent Method for Network Calibration

The calibration method for finding r_1, x_1 of a network as described in Section III involves at least three measurements of input impedance (or reflection coefficient) and corresponding measurements of lengths d of shorted transmission lines, which may offer contributions to errors in the evaluation of r_1 and x_1. It is now to be shown that one of the greatest advantages of the impedance transformation method is that significant errors may be minimized through a plot of x_L' vs x_L, which is seen to be a straight line from (6). With the output consecutively terminated in a sufficient number of shorted transmission-line sections, a mean straight line drawn through the experimental points serves to eliminate the largest errors which are likely to occur. We shall now elaborate on this technique.

The cotangent method may be developed with reference to Fig. 3. Let 3-4 represent the output port plane, while 1-2 represents the position of the input standing-wave voltage minimum when the output is shorted at 3-4. Now connect a quarter-wavelength line extension to 3-4 so that the short is then located at 3′-4′, and such that the input voltage minimum has shifted toward the load to 1′-2′. Let d_1 be the standing wave shift from 1′-2′ measured positive toward the generator.

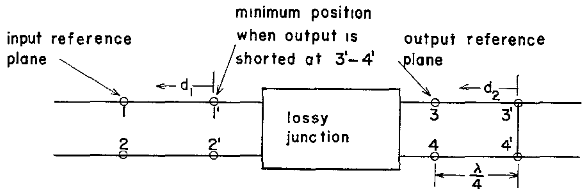


Fig. 3—Location of references and measurement of distances used in the cotangent method.

Recall that when the short is placed at 3′-4, $x_L' = (x_L + x_1)/r_1 = \infty$ and the corresponding input reflection coefficient establishes point D on the Γ_{in} plane of Fig. 2(a). Now vary distance d_2 measured from 3′-4′ by connecting several shorted transmission lines successively to 3-4, such that $-\lambda/4 \leq d_2 \leq \lambda/4$. The corresponding Γ_{in} is measured for each d_2 so that the circular Γ_{in} plot of Fig. 2(a) may be obtained.

Now from the discussion of Section III, any angle EAD in the Γ_{in} plane transforms into angle $E'A'D'$ in the Γ_L' plane of Fig. 2 (b). Let us define an equivalent distance d_1' (Fig. 2) such that angle $EAD = 2\,kd_1'$. The Γ_L' plane reference is taken at $\Gamma_L' = +1$, so $x_L' = \cot kd_1'$ and $x_L = \cot kd_2$. Substituting x_L' and x_L into (6) yields

$$\cot kd_1' = \frac{1}{r_1}\cot kd_2 + \frac{x_1}{r_1}. \tag{11}$$

The desired calibration constants r_1, x_1 can evidently be found from (11) by noting that the slope of the line is $1/r_1$ while the ordinate intercept is $-x_1$. We note in passing that when the junction is lossless, then $d_1' = d_1$, equation (11) becomes the same as that which was proposed by Oliver[9] for lossless structures.

A sample plot of data and calibration curve for a typical coaxial line junction is shown in Figs. 4 and 5. A sample input impedance measurement was plotted as point F on Fig. 4. The radius of the Γ_{in} circle is 0.908, the angle FAD measures 60.8°, and line AF has a length of 0.368, so

$$\overline{A'F'} = \frac{\overline{AF}}{\overline{AD}}\,\overline{A'D'} = 0.394\,e^{j60.8°}.$$

The modified load impedance is read out as $z_L' = 1.10 + j0.90$ and, by applying (6), we find $r_L = 0.844$ and $x_L = 0.316$.

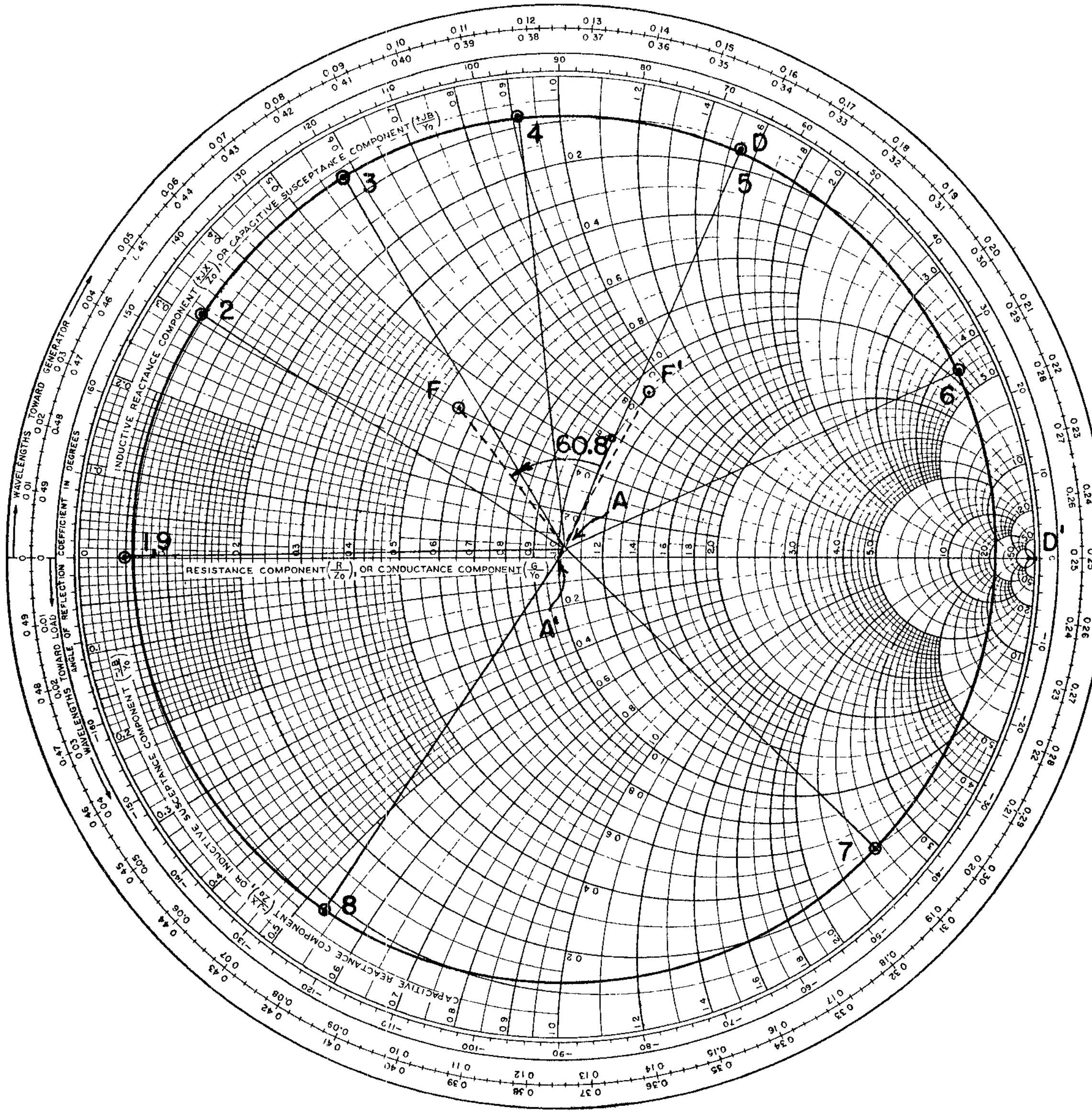


Fig 4—Sample plot of calibration measurements. Γ_{in} and $\Gamma_{L'}$ are plotted on the same plane.

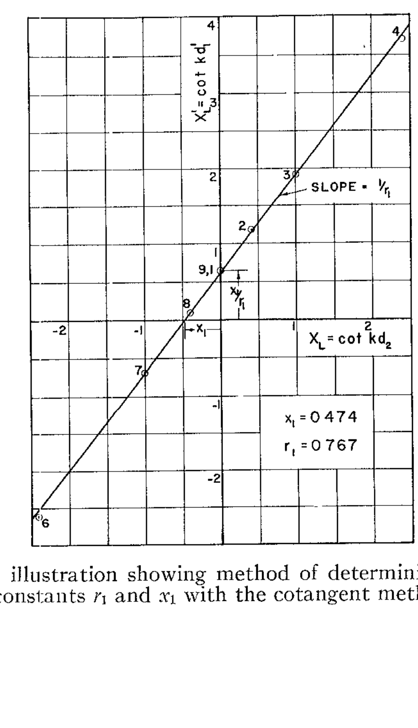


Fig. 5—An illustration showing method of determining calibration constants r_1 and x_1 with the cotangent method.

V. General Discussion of Calibration Procedure

The measurement procedure described involves measurement of the input reflection coefficient for various values of output terminating reactace obtainable by terminating the port 3-4 with a transmission line equipped with a sliding short. The input reflection coefficient can usually be measured with a slotted line; however, other methods may be employed depending upon the accuracy desired, the nature of the magnitude of Γ_{in}, and the operating wavelength.

A minimum of three different terminating reactances are needed to locate the circle in the Γ_{in} plane. If the calibration constants are to be determined by (10), these three reactances correspond to short-circuited line lengths d_2 of zero, $\lambda/8$ and $\lambda/4$ (or terminating impedances of $j0$, $j1$ and $j\infty$), respectively. Of course, more points would provide more accuracy for determining the center of the Γ_{in} circle. If the cotangent method of evaluating the calibration constants is to be used, then several points should be obtained so that a mean straight line can be drawn.

After collecting enough data to plot the Γ_{in} circle to the desired accuracy, one can then locate its center A, shown typically in Fig. 2(a). The distance AD corresponds to $A'D'$ in the Γ_L' plane, and any Γ_L' may then be determined by the relations

$$\frac{AC}{AD} = \frac{A'C'}{A'D'} \text{ and angle } CAD = \text{ angle } C'A'D'.$$

It should be mentioned at this point that if the z matrix is *positive real*,[15] Γ_{in} will always lie within the circle DED and the junction will be passive.

Appendix I

Proof of Linear Transformation Between Γ_{in} and Γ_L' Planes

The complex constants defined by (8) are easily representable as vectors which are shown in Fig. 6. From (9), when $\Gamma_L' = +1$, $\Gamma_{in} = \bar{a}$ and when $\Gamma_L' = 0$, $\Gamma_{in} = \bar{a} + \bar{b}$ so $\bar{a}$ equals vector $\overline{OD}$ and $\bar{b}$ equals vector $\overline{DA}$.

The linear transformation property may be seen by letting Γ_L' take on any value such that Γ_{in} takes on a corresponding value, say vector $\overline{OF}$. Eq. (9) may be rearranged to read $(\bar{a}+\bar{b}) - \Gamma_{in} = \bar{b}\Gamma_L'$. It follows then that vector $\overline{FA} = \bar{b}\Gamma_L'$, and A maps into A' by merely letting $\Gamma_L' = 0$. Also, taking a ratio of the two vectors $\bar{b}\Gamma_L'$ and $\bar{b}$ yields

$$\frac{\bar{b}\Gamma_L'}{\bar{b}} = \frac{\overline{FA}}{\overline{DA}} = \Gamma_L'. \tag{12}$$

[15] The necessary and sufficient condition for positive reality is that $R_{11}R_{22} - R_{12} \geq 0$ where R_{11}, R_{22}, and R_{12} are the real parts of Z_{11}, Z_{22}, and and Z_{12}, respectively. See E. A. Guillemin, "Communication Networks," John Wiley and Sons, Inc., New York, N. Y., vol. 2, p. 216; 1935.

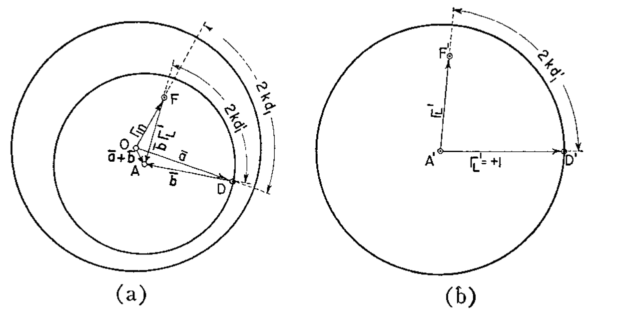


Fig. 6—Γ_{in} plane showing complex constants $\bar{a}$ and $\bar{b}$. (a) Γ_{in} plane. (b) $\Gamma_{L'}$ plane.

Thus, the angle $FAD = 2kd_1 =$ angle $F'A'D'$ and the magnitude of Γ_L' is found through equal magnification of the radii

$$\left|\frac{FA}{AD}\right| = \left|\frac{F'A'}{A'D'}\right|. \tag{13}$$

The validity of the proposed point-for-point transformation is thereby established.

Appendix II

Transformation Method Relationship to Network Impedances and Scattering Coefficients

It may be convenient to use the transformation procedure outlined in the text to determine the impedance constants of the T-network shown in Fig. 1 or the commonly used scattering coefficients, both of which completely describe the junction. These are easily found through the use of the constants $\bar{a}$ and $\bar{b}$ defined by (8) and the calibration constant $z_1 = r_1 + jx_1$ (see Appendix I).

A. Evaluation of the Network Impedance

Using the first of (8) it is apparent that

$$z_{11} = \frac{Z_{11}}{Z_{01}} = \frac{1+\bar{a}}{1-\bar{a}}. \tag{14}$$

The complex constant $\bar{b}$ may be written as

$$\bar{b} = \frac{-Z_{01}}{Z_{02}r_1}\left(\frac{Z_{12}}{Z_{11}+Z_{01}}\right)^2. \tag{15}$$

After eliminating $Z_{12}^2/(Z_{11}+Z_{01})$ from (4) with (15) and subsituting (14) we find .

$$z_{22} = \frac{Z_{22}}{Z_{02}} = z_1 - \frac{2\bar{b}r_1}{1-\bar{a}}. \tag{16}$$

Finally, z_{12} may be found by substituting (14) and (16) into (4) yielding

$$z_{12}^2 = \frac{Z_{12}^2}{Z_{02}Z_{01}} = \frac{-4\bar{b}r_1}{(1-\bar{a})^2}. \tag{17}$$

B. Evaluation of the Scattering Coefficients

The scattering coefficients are often needed to describe the characteristics of a network. For our discussion we shall assume that the junction is reciprocal as before, *i.e.*, $S_{12}=S_{21}$.

To find the forward reflection coefficient S_{11} we note that when $\Gamma_L=0$, then $r_L=1$ and $x_L=0$. Then, from (6) and (7),

$$\Gamma_L' = \frac{z_L'-1}{z_L'+1} = \frac{1-r_1+jx_1}{1+r_1+jx_1} = -\frac{z_1^*-1}{z_1+1}, \qquad (18)$$

where z_1^* denotes the complex conjugate of z_1. Substituting (18) into (9) and rearranging, we obtain

$$\Gamma_{in} = \bar{a} + \frac{2\bar{b}r_1}{z_1+1}, \qquad (19)$$

but recall that the input reflection coefficient may also be expressed in terms of the scattering coefficients of the network, or

$$\Gamma_{in} = S_{11} + \frac{S_{12}^2\Gamma_L}{1-S_{22}\Gamma_L}. \qquad (20)$$

Letting $\Gamma_L=0$, we see that

$$S_{11} = \bar{a} + \frac{2\bar{b}r_1}{z_1+1}. \qquad (21)$$

It follows from (4) and the above discussion that

$$S_{22} = \frac{z_1-1}{z_1+1}. \qquad (22)$$

The transmission coefficient S_{21} may be found by observing that when $\Gamma_L=+1$ then $\Gamma_L'=+1$ and $\Gamma_{in}=\bar{a}$, or

$$\Gamma_{in} = S_{11} + \frac{S_{12}^2}{1-S_{22}}. \qquad (23)$$

Substituting (21) and (22) into (23) yields

$$S_{12}^2 = \frac{-4\bar{b}r_1}{(z_1+1)^2}. \qquad (24)$$